Math 255A Lecture 6 Notes

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1 Weak Solutions of the Poisson Equation and Strengthened Hahn-Banach

1.1 Weak solutions of the Poisson equation

Last time, we were trying to solve the equation $\Delta u = f$ for $f \in L^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open and bounded.

Proposition 1.1. *There exists a constant $A > 0$ such that for any $\varphi \in C_0^2(\Omega)$ (C^2 functions on Ω with compact support), we have*

$$\|\varphi\|_{L^2(\Omega)} \leq A\|\Delta\varphi\|_{L^2(\Omega)}.$$

Proof. For simplicity of notation, we assume φ is real. Then, using integration by parts,

$$\int_\Omega \varphi\Delta\varphi\,dx = \sum_j \int_\Omega \varphi\frac{\partial^2\varphi}{\partial x_j^2}\,dx = -\sum_j \int_\Omega \left(\frac{\partial\varphi}{\partial x_j}\right)^2\,dx = -\int_\Omega |\nabla\varphi|^2\,dx.$$

Also,

$$\int_\Omega x_1 \underbrace{2\varphi\frac{\partial\varphi}{\partial x_1}}_{=\partial_{x_1}(\varphi^2)}\,dx = -\int_\Omega \varphi^2\,dx$$

implies that, using Cauchy-Schwarz,

$$\|\varphi\|_{L^2}^2 \leq 2C\int_\Omega |\varphi||\partial_{x_1}\varphi|\,dx \leq 2C\|\varphi\|_{L^2}\|\nabla\varphi\|_{L^2}.$$

Thus,

$$\|\varphi\|_{L^2} \leq 2C^2\|\Delta\varphi\|_{L^2}.$$

□

Now let $\Delta C_0^2(\Omega) = \{\Delta\varphi : \varphi \in C_0^2(\Omega)\} \subseteq L^2(\Omega)$. Consider the linear form $L : \Delta C_0^2(\Omega) \to \mathbb{C}$ that sends $\Delta\varphi \mapsto \int_\Omega f\varphi\, dx$, where $f = \Delta u$. This form L is well-defined (thanks to the proposition), and we get

$$|L(\Delta\varphi)| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le 4C^2\|f\|_{L^2}\|\Delta\varphi\|_{L^2}.$$

By Hahn-Banach, L extends to a continuous linear form $\tilde{L}$ on all of $L^2(\Omega)$ such that

$$|\tilde{L}(v)| \le 4C^2\|f\|_{L^2}\|v\|_{L^2}$$

for any $v \in L^2(\Omega)$. By the Riesz representation theorem, there exists $u \in L^2(\Omega)$ such that $\tilde{L}(v) = \int_\Omega vu\, dx$ for any $v \in L^2$ and $\|u\|_{L^2} \le 4C^2\|f\|_{L^2}$.

When $v = \Delta\varphi$ with $\varphi \in C_0^2(\Omega)$, we get

$$\int_\Omega u\Delta\varphi\, dx = \int_\Omega f\varphi\, dx.$$

If u were of class $C^2(\Omega)$, we would get $\int \Delta u\varphi = \int f\varphi \ \forall\varphi$, so $\Delta u = f$ a.e. In general, $u \in L^2(\Omega)$ satisfying the above equation is called a **weak solution** of $\Delta u = f$.

1.2 Strengthened Hahn-Banach theorem

Theorem 1.1. *Let V be a normed vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let $T : V \to V$ be a continuous linear map such that $\|T\| \le 1$ ($\|Tx\| \le \|x\|$ for all $x \in V$). Assume that T has a fixed point $x_0 \ne 0$ such that $Tx_0 = x_0$. Then there is a linear continuous form $f : V \to K$ such that $\|f\| = 1$, $f(x_0) = \|x_0\|$, and $f(Tx) = f(x)$ for all $x \in V$.*

Proof. Let us define $\|x\|_T = \inf \|\sum_{n=0}^\infty \lambda_n T^n x\|$, where the inf is taken over all $\lambda_n \ge 0$ such that $\sum \lambda_n = 1$, where only finitely many are nonzero in this sum.[1] We claim that $x \mapsto \|x\|_T$ is a seminorm on V. We only need to check the triangle inequality. Let $x, y \in V$ and $\varepsilon > 0$. Then there exist $\lambda_n \ge 0$ and $\mu_n \ge 0$ with $\sum \lambda_n = \sum \mu_n = 1$ such that

$$\left\|\sum \lambda_n T^n x\right\| < \|x\|_T + \varepsilon, \qquad \left\|\sum \mu_n T^n y\right\| < \|y\|_T + \varepsilon.$$

By the triangle inequality,

$$\begin{aligned}\left\|\left(\sum \lambda_n T^n\right)\left(\sum \mu_n T^n\right)(x+y)\right\| &\le \left\|\left(\sum \lambda_n T^n\right)\underbrace{\left(\sum \mu_n T^n\right)}_{\text{norm}\le 1} x\right\| \\ &\quad + \left\|\underbrace{\left(\sum \lambda_n T^n\right)}_{\text{norm}\le 1}\left(\sum \mu_n T^n\right) y\right\|\end{aligned}$$

[1] This is a really clever choice for a seminorm.

$$\leq \|x\|_T + \|y\|_T + 2\varepsilon.$$

Now observe that

$$\|x+y\|_T \leq \left\| \sum \left(\sum_{n+m=j} \lambda_n \mu_m \right) T^j (x+y) \right\| = \left\| \left(\sum \lambda_n T^n \right) \left(\sum \mu_n T^n \right) (x+y) \right\|.$$

Apply Hahn-Banach with respect to this seminorm. Let $g : Kx_0 \to \mathbb{C}$ send $\alpha x_0 \mapsto \alpha \|x_0\|$. Then $|g(y)| = \|y\| = \|y\|_T$ for all $y \in Kx_0$. Then g extends to a linear form f such that $f(x_0) = \|x_0\|$ and $|f(x)| \leq \|x\|_T$ for $x \in V$. Finally, check that $f(Tx) = f(x)$:

$$|f(Tx) - f(x)| = |f(Tx - x)| \leq \|Tx - x\|_T = 0$$

for all x, where the last equality comes from

$$\left\| \frac{1}{N}(1 + T + \cdots + T^{N-1})(Tx - x) \right\| = \left\| \frac{1}{N}(T^N x - x) \right\| \leq \frac{2\|x\|}{N} \xrightarrow{N\to\infty} 0. \qquad \square$$

Remark 1.1. When T is the identity, this is the usual Hahn-Banach theorem.

1.3 Generalized Banach limits

Here is an application due to Banach himself. Let $V = \ell^\infty(\mathbb{N})$ with elements $x = (x_1, x_2, \ldots,)$ with $x_j \in \mathbb{C}$. Let the shift operator be $(Tx)_j = x_{j+1}$. By the theorem, there is a continuous linear form $f : \ell^\infty \to \mathbb{C}$ such that $\|f\| = 1$, $f(1, 1, \ldots) = 1$, $f(Tx) = f(x)$ for all $x \in \ell^\infty$. Note that

$$|f(x)| = |f(T^n x)| \leq \sup_{j>n} |x_j|,$$

so

$$|f(x)| \leq \limsup_{n\to\infty} |x_n|.$$

For all $c \in \mathbb{C}$ plugging in $x + (c, c, \ldots)$ gives

$$|f(x) - c| \leq \limsup_{n\to\infty} |x_n - c|.$$

So if (x_n) converges, then $f(x) = \lim_{n\to\infty} x_n$.

Remark 1.2. If x is a real sequence, $\liminf x_n \leq f(x) \leq \limsup x_n$.